Stability and thermodynamics of black rings

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We study the phase diagram of $D = 5$ rotating black holes and the black rings discovered by Emparan and Reall. We address the issue of microcanonical stability of these spacetimes and its relation to thermodynamics by using the so-called "Poincaré method" of stability. We are able to show that one of the BR branches is always unstable, with a change of stability at the point where both BR branches meet. We study the geometry of the thermodynamic state space ("Ruppeiner geometry") and compute the critical exponents to check the corresponding scaling laws. We find that, at extremality, the system exhibits a behaviour which, formally, is very similar to that of a second order phase transition.

1 Introduction

Our aim in the present talk is to report on investigations on the properties of the phase diagram of rotating BH solutions (with just one angular momentum J) of $D = 5$ gravity. This scenario seems appropriate to us due to the discovery made by Emparan and Reall of a new BH phase: a rotating BH with event horizon topology $S^1 \times S^2$ — what they called a "black ring" [1]. A crucial property of these black rings is that they exist in a region of parameter space (the total mass M and angular momentum J) which overlaps with that of the spherical black hole, thus providing the first known example of BH non-uniqueness in asymptotically flat space. This raises questions on stability and possible transitions between the different phases, which is what we wish to consider here. The present contribution is based on Ref. [2].

Black Holes and Black Rings

The five dimensional rotating black hole [3] has an upper "Kerr bound" in its angular momentum per unit mass, the configuration saturating the bound being a naked singularity. In the following we will use the dimensionless quantity

$$x \equiv \sqrt{27\pi/32G}\, J/M^{3/2} \tag{1}$$

as the "control parameter"[3] of the problem — analogous, say, to the temperature of a liquid-gas system in the canonical ensemble. The rotating BH solution exists in the range of parameter space:

$$\text{Black Hole}: \qquad 0 \le x < 1\,,$$

On the other hand, one can have *two* different black ring spacetimes with horizon topology $S^1 \times S^2$ when x exceeds a certain minimal value given by $x_{\rm min} = \sqrt{27/32} \approx 0.92$. One of

[3] A proper *order* parameter will be defined below.

them, which we will call the "large" BR, can have an angular momentum per unit mass unbounded from above. The other one, that we call "small" BR, cannot. One has:

$$\begin{cases} \text{Large Black Ring}: & x_{\min} \le x < \infty\,, \\ \text{Small Black Ring}: & x_{\min} \le x < 1\,. \end{cases}$$

The entropies are plotted in Fig. 1. Near extremality the LBR becomes entropically favoured. This was taken in [1] as an indication of a phase transition as we vary x.

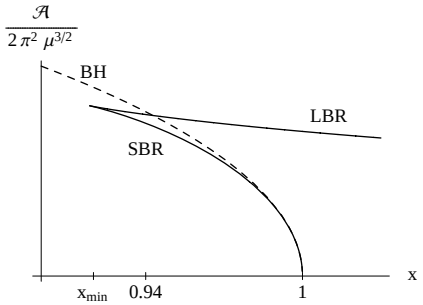

Fig. 1. A suitably normalised ($\mu \equiv 8GM/3\pi$) area of the horizon of the three phases as a function of x. At $x = 2\sqrt{2}/3 \approx 0.94$ the entropy of the large BR exceeds that of the BH.

2 Dynamical vs. Thermodynamical Stability

The first issue we want to address is that of the stability of the classical solutions. To face this problem, a way to circumvent the study of linear perturbations of the metric might come from thermodynamics. In ordinary thermodynamics of extensive systems, local *thermodynamical* stability (defined as the Hessian of the entropy having no positive eigenvalues) is linked to the *dynamical* stability of the system: a positive mode in the Hessian means that at least some kind of small fluctuations within the system are entropically favoured, implying that the system is unstable against those. However, the simple argument [4] leading to this conclusion relies on the *additivity* of the entropy — a property which does not hold for black holes (or self-gravitating systems in general) because these cannot be thought as made up of constituent subsystems. The best known example of this failure is the Schwarzschild black hole: a stable configuration with negative specific heat (i.e. a positive Hessian).

To avoid these problems, we will propose here to use the so-called "Poincaré" [5] or "turning point" method of stability, in the way that it was first applied by Katz [6] for the study self-gravitating systems. This method does not require additivity of the entropy function, being based solely on entropy maximisation. It was applied to BHs for the first time in [7] and, contrary to the standard criteria based on the signs of the specific heats, it does *not* predict any instability in e.g. the Schwarzschild or Kerr metrics.

We want to study the stability of solutions describing isolated BHs. The appropriate thermodynamic ensemble is thus the microcanonical and the relevant potential is the entropy. The crucial point is to realize that a "fundamental relation" of a BH like $S = S(M, J)$ only contains information about the *equilibrium* BH-series, but any discussion based on the entropy maximum principle requires the knowledge of the behaviour of the entropy *off* equilibrium. For a nonextensive system, such behaviour will be in general different from the one given by $S(M, J)$. What one needs is then an "extended entropy function", denoted here by $\widehat{S} = \widehat{S}(X^\rho; M, J)$, defined also for non-equilibrium states described by some set of

off-equilibrium variables $\{X^\rho\}$[4]. Equilibrium configurations are those which obey:

$$\partial_\rho \widehat{S} \equiv \partial \widehat{S}/\partial X^\rho = 0\,, \tag{2}$$

i.e. the extrema of the entropy at constant M and J. They are described by $X^\rho_{\rm eq} = X^\rho_{\rm eq}(M,J)$, the solutions to Eq. (2) . $\widehat{S}$ gives back the (known) equilibrium value S when evaluated at $X^\rho_{\rm eq}$, i.e $S = \widehat{S}(X^\rho_{\rm eq}; M, J)$. A given equilibrium configuration is *stable* if the eigenvalues

$$\lambda_\rho(M,J) \equiv (\partial_{\rho\rho}\widehat{S})_{\rm eq} \tag{3}$$

are all negative, and unstable otherwise[5]. In general, the function $\widehat{S}$ is completely unknown. However, one can infer something about its behaviour by plotting the appropriate phase diagram along the *equilibrium* series. Defining the off-equilibrium variable conjugate to J:

$$\widehat{\omega} \equiv \partial_J \widehat{S}\,, \tag{4}$$

which reduces to $\omega \equiv \partial_J S$ when evaluated at equilibrium (i.e. $\omega = (\widehat{\omega})_{\rm eq}$), one can show that[6] [6]:

$$\partial_J \omega = (\partial_J \widehat{\omega})_{\rm eq} - \sum_\rho (\partial_\rho \widehat{\omega})^2_{\rm eq}/\lambda_\rho\,. \tag{5}$$

A change of stability takes place when one or several λ_ρ approach zero and change sign. Near such a point, the term(s) proportional to $\sim 1/\lambda_\rho$ will dominate in Eq. (5), and therefore the equilibrium "conjugacy diagram" $\omega = \omega(J)$ will exhibit a vertical tangent[7]. This is often referred to as a "turning point". The branch with a negative slope *near* a turning point is always unstable, since for $\partial_J\omega \to -\infty$ at least one of the vanishing λ_ρ in (5) has to be positive. On the other hand, if we do not know about the stability of at least one particular point of the positive-slope (near the turning point) branch, then we cannot say anything about its full stability, but only that it is "more stable" than the negative-slope branch (since it will have at least one unstable mode less). Also, there might be positive eigenvalues that never change sign, and therefore these will never show up in a conjugacy diagram.

A fundamental theorem in stability theory [8] shows in fact that the stability of a linear series of equilibria can change *only* at a turning point or at a bifurcation (i.e. when a given equilibrium series intersects with another one). Note that a change in the sign of the specific heat takes place at a horizontal tangent, but in nonextensive thermodynamics this has no relation to any change of stability.

The plot of the conjugacy diagram $\omega = \omega(J)$ for the BH and BR phases is shown in Fig. 2. We see no turning points (i.e. no changes of stability) along the BH branch. Given the fact that one point in the curve (namely, the Schwarzschild limit) is stable [9], this means that the $D = 5$ rotating BH is also stable *unless* 1) there are other equilibria that bifurcate from the BH-series or 2) there are unstable modes that this method does not probe. As for the BRs, we see a turning point at $x_{\rm min}$. This automatically implies that the SBR is *locally* unstable. The LBR branch has to be more stable than the SBR, although this does not prove its full dynamical stability. In fact, at least the large-x limit of the LBR is expected to suffer from black-string-like instabilities [10]. It would be interesting to see if such instabilities actually appear as a bifurcation (a "non-uniform ring branch", say) at some value of x.

Although not shown in this plot, let us mention that along the BH branch (at $x = \frac{1}{2}$) there is a change in the sign of the specific heat at constant J [2], exactly analogous to the $D = 4$ case [11]. However, this point is a horizontal tangent in the plot of $\beta(M) = \partial_M S$, and therefore, according to this analysis, no changes of stability are associated to it.

[4] They should be thought as metric perturbations $X^\rho \sim \delta g^{(\rho)}_{\mu\nu}$, since there are no other possibilities for coordinates in configuration space.

[5] We take off-equilibrium coordinates $\{X^\rho\}$ such that the matrix $(\partial_{\rho\sigma}\widehat{S})_{\rm eq}$ is diagonal.

[6] Obviously, any pair of conjugate variables (like e.g. $\beta(M) = \partial_M S$) will do as well.

[7] Unless also $(\partial_\rho \widehat{\omega})_{\rm eq} = 0$. In such a case one can show [8] that there is a *bifurcation* — see below.

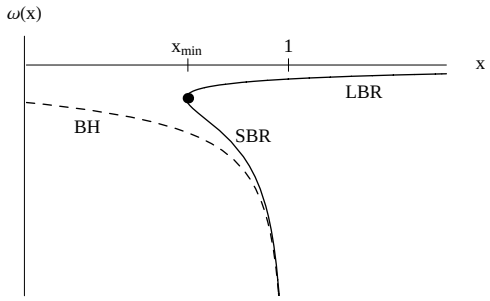

Fig. 2. $\omega = \partial_J S$ as a function of x (i.e. J at fixed mass). There is a turning point at $x_{\rm min}$. The stability of black rings changes there, while no changes of stability appear in the BH branch.

3 Thermodynamic Geometry

It has been observed by several authors in different contexts that the entropy function induces a natural metric on the thermodynamic state space, and that the geometric invariants constructed out of it provide information about the phases of the model under consideration. This formalism was pioneered by Ruppeiner [12]. The Ruppeiner metric is defined by (minus) the second derivatives of the entropy. In our off-equilibrium formalism, where we have $\widehat{S} = \widehat{S}(X^\rho; M, J) \equiv \widehat{S}(X^\rho; \mu^i)$, this means:

$$g_{ij} = -(\partial_{ij}\widehat{S})_{\rm eq}\,, \qquad g_{i\rho} = -(\partial_{i\rho}\widehat{S})_{\rm eq}\,, \qquad g_{\rho\rho} = -\lambda_\rho\,. \tag{6}$$

Using the standard techniques of thermodynamic fluctuation theory [12, 13], from the quadratic expansion of $\widehat{S}$ one can compute the second moments of correlations of any quantities around equilibrium. These turn out to be given by the inverse elements of the Ruppeiner metric. In particular:

$$\langle \Delta\mu^i \Delta\mu^j \rangle = g^{ij}\,. \tag{7}$$

A most important quantity in Ruppeiner theory is the curvature scalar R of this metric. Among other properties, calculations show [12] that R diverges at a critical point, and that it does so in the same way as the correlation volume:

$$R_{\rm crit} \sim \xi^d\,, \tag{8}$$

where ξ is the correlation length and d the effective dimension of the system.

The Ruppeiner metric as defined above cannot be computed explicitly, since in general we have no knowledge about the off-equilibrium entropy function $\widehat{S}$. However, it can be shown [2] that near $x_{\rm min}$ (turning point) in the LBR branch, and near $x = 1$ (vertical asymptote) in the BH/SBR branches, the quadratic fluctuations in β and ω (the variables conjugate to M and J) diverge and, furthermore, that they can be well approximated by the *equilibrium* Hessian (which in fact does diverge at those points — see [2] for the explicit computation). Using (7) it can be shown that:

$$g^{ij} \approx \pm\partial_{ij} S\,, \tag{9}$$

where the plus sign stands for the LBR branch near $x_{\rm min}$ while the minus sign stands for the BH and SBR branches near $x = 1$. That is, *near* $x_{\rm min}$ and $x = 1$, the "effective" elements of the Ruppeiner metric are just a few and, moreover, they can be computed from the equilibrium entropy.

We have computed the thermodynamic curvature for the BH/BR system and the result is plotted in Fig. 3. For convenience, the plot shows the values of R as computed from (9) (with the plus sign) for all x. One can see that not only the metric, but also the thermodynamic curvature scalar diverges at $x_{\rm min}$ and $x = 1$.

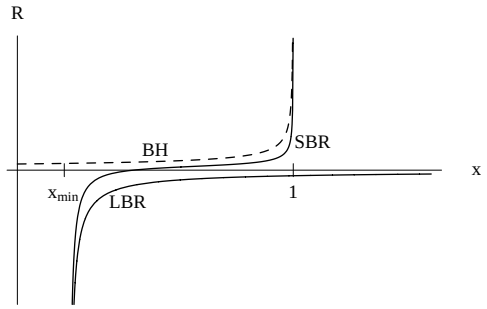

Fig. 3. Thermodynamic curvature scalar. The relevant values are those near $x = 1$ and $x_{\rm min}$.

4 Critical Exponents

Finally, we turn now to the study of the scaling relations familiar from RG-theory in Statistical Mechanics at the points $x_{\rm min}$ and $x = 1$. We first define the appropriate susceptibilities and critical exponents that are suited to the microcanonical ensemble [14]:

$$\chi_J \equiv \partial_M^2 S \sim \epsilon_M^{-\alpha}\,,\ \epsilon_J^{-\varphi}\,, \qquad \chi_M \equiv \partial_J^2 S \sim \epsilon_M^{-\gamma}\,,\ \epsilon_J^{-\sigma}\,. \tag{10}$$

These are the natural generalisations of the expressions familiar from other ensembles. At $x_{\rm min}$ and $x = 1$ the parameters ϵ_M and ϵ_J are defined to be, respectively:

$$\epsilon_M = \tfrac{M_{\rm min}-M}{M_{\rm min}}\,,\ \epsilon_M = \tfrac{M-M_{\rm ext}}{M_{\rm ext}}\,; \qquad \epsilon_J = \tfrac{J-J_{\rm min}}{J_{\rm min}}\,,\ \epsilon_J = \tfrac{J_{\rm ext}-J}{J_{\rm ext}}\,. \tag{11}$$

Other critical exponents tell us about the behaviour of the order parameter η of the transition considered. In general, there is no obvious choice for such order parameter. Kaburaki proposed in [14] to choose, as an order parameter in BH phase transitions, the difference of the conjugate variables between the two phases. In our case this means:

$$\text{At } x_{\rm min}\text{:}\ \ \eta \equiv \omega_{\rm SBR} - \omega_{\rm LBR}\,. \qquad \text{At } x = 1\text{:}\ \ \eta \equiv \omega_{\rm SBR} - \omega_{\rm BH}\,. \tag{12}$$

With these definitions, the critical exponents β and δ familiar from statistical mechanics are given by:

$$\eta \sim \epsilon_M^{\beta} \sim \epsilon_J^{\delta^{-1}}\,. \tag{13}$$

All these critical exponents are readily computed to be:

$$\text{At } x_{\rm min}\text{:}\ \ \alpha,\varphi,\gamma,\sigma = \tfrac{1}{2}\,;\ \beta,\delta^{-1} = \tfrac{1}{2}\,. \qquad \text{At } x = 1\text{:}\ \ \alpha,\varphi,\gamma,\sigma = \tfrac{3}{2}\,;\ \beta,\delta^{-1} = -\tfrac{1}{2}\,. \tag{14}$$

The scaling relations involving the critical exponents defined so far are given by [13]:

$$\alpha + 2\beta + \gamma = 2\,, \qquad \beta(\delta - 1) = \gamma\,, \qquad \varphi(\beta + \gamma) = \alpha\,. \tag{15}$$

One can check that these scalings are satisfied both at $x = x_{\rm min}$ and at $x = 1$.

However, at a critical point of a second order phase transition further scaling laws involving other critical exponents are satisfied. The remaining critical exponents include those related to the behaviour of the correlation length, whose divergence at a critical point is expressed in terms of the exponents ν and μ as:

$$\xi \sim \epsilon_M^{-\nu} \sim \epsilon_J^{-\mu}\,, \tag{16}$$

which obey the scaling laws [13]:

$$2 - \alpha = \nu d\,, \qquad \mu(\beta + \gamma) = \nu\,. \tag{17}$$

Even if there is no obvious way to compute (or even define) ξ for a BH geometry, Eq. (8) provides us with an explicit algorithm to compute νd and μd once we have computed the divergence of the thermodynamic curvature scalar. The result is [2]:

$$\text{At } x_{\min}\text{:}\ \ \nu d = \mu d = 1\,. \qquad \text{At } x = 1\text{:}\ \ \nu d = \mu d = \tfrac{1}{2}\,. \tag{18}$$

Relations (17) are obeyed at $x = 1$ for any effective d, but they are not satisfied at $x_{\min}$.

5 Conclusions

Using the Poincaré method of stability we have shown that the SBR is unstable not only globally (see Fig. 1) but also locally. Stability changes at $x_{\min}$, where Fig. 2 exhibits a turning point. Near $x = 1$ the BH and SBR branches behave in a way similar to that of a system near a critical point. However, we cannot speak about a second order phase transition here, since at least one phase (the SBR) is unstable. Also, we do not know about any phases beyond the "critical point". The found scaling behaviour has to be interpreted then as describing, formally, the properties of the system *only* along the stable directions in configuration space. However, we find such a property nontrivial. Finally, nothing special happens at the point in the BH branch where the specific heat changes sign ($x = \frac{1}{2}$) or at $x = 2\sqrt{2}/3 \approx 0.94$ (see Fig. 1). Given this fact, *if* both the BH and LBR are stable around $x \approx 0.94$, the only possibility we are left with is that of a first order phase transition between both. It is also interesting how the thermodynamic curvature seems to reproduce the behaviour of the correlation length ξ. However, without an explicit computation of ξ for these BH geometries, Eq. (8) is just an Ansatz for it. It would be desirable to understand this in a better way.

Acknowledgments

E.L.-T. wants to thank the organizers of the XXVII Spanish Relativity Meeting for a successful and interesting conference and for their kind invitation to give this talk, as well as to the participants J. L. F. Barbón, D. Cremades, R. Emparan, R. Myers and J. Russo for their interesting questions and comments on the present work. This work has been supported in part by a Marie Curie Fellowship under contract MEIF-CT-2003-502349 and the Spanish Grant BFM2003-01090.

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